Math 1A: Calculus **Instructor: Alexander Paulin**

Handout: Curve Sketching / L'Hôpital's Rule

Discussions 201, 203 // 2018-10-24

1. Curve Sketching

Problem 1. Without using a graphing calculator, sketch the graph of

$$f(x) = \frac{x^2+5}{x-2}.$$

Your picture does not need to be quantitatively accurate, but it should be qualitatively close. Label the intercept(s), asymptote(s), and local maxima/minima.

What is the domain of f? What is the range of f?

Problem 2. Consider the function given by

$$g(x) = (x+2)^{1/3}.$$

Find all (if there are any) of the following: (a) critical points (b) local maxima and minima (c) inflection points.

Problem 3. Suppose that $h(x)$ is a *cubic polynomial*. Prove that h has exactly one inflection point.

Problem 4 (Adapted from Stewart §4.4, Ex. 85)**.** As in the preceding problem, suppose that $h(x)$ is a cubic polynomial. Moreover, suppose that there are three real numbers $s_1 < s_2 < s_3$ for which

$$h(s_1) = h(s_2) = h(s_3).$$

Prove that the x-coordinate of the unique inflection point of h (see preceding problem) is equal to

$$\frac{s_1+s_2+s_3}{3}.$$

2. L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule)**.** *Let f, g be functions, and fix c to be either a real number or one of the symbols $-\infty, \infty$. We are interested in evaluating the limit*

$$\lim_{x\to c} \frac{f(x)}{g(x)}.$$

Suppose that

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \quad \textit{or} \quad \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty.$$

(That is to say, our limit is in one of the appropriate "indeterminate forms.")

Then, ***if the limit*** $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ ***exists OR is*** $\pm\infty$*, we can conclude*

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.}$$

You do not need to remember all the details of the rule, but **please** remember to at least check that your limit is one of the indeterminate forms where L'Hôpital's is applicable before using it blindly...

Sometimes your limit may not be presented to you as a fraction, but L'Hôpital's can still be useful. You just need to find a way to rewrite the expression as a fraction first.

Problem 5. On the first midterm, many people claimed that

$$\lim_{x\to 0} x \ln x = 0$$

but weren't able to prove it. Now you have the tools to do so (use L'Hôpital's rule).

Problem 6. Evaluate the limit

$$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}.$$

Problem 7. Try to use L'Hôpital's rule to evaluate the limit

$$\lim_{x\to\infty} \frac{x + \sin(x)}{x}.$$

What happens? Figure out some other way to evaluate this limit.

Problem 8. The limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

can be evaluated very easily using L'Hôpital's rule. But it would be a *faux pas* to do so. Why is that?